Chapter 3

Oscillations

In this chapter we will discuss oscillatory motion. The simplest examples of such a motion are a swinging pendulum and a mass attached to the end of a spring, but it is possible to make a system more complicated by introducing a damping force and/or an external driving force. We will study all of these cases.

We are interested in oscillatory motion for two reasons. First, we study it because we *can* study it. This is one of the few systems in physics where we can solve for the motion exactly. (There's nothing wrong with looking under the lamppost every now and then.) Second, such systems are ubiquitous in physics, for reasons that will become clear in Section 4.2. If there was ever a type of physical system worthy of study, this is it.

We'll jump right into some math at the beginning of this chapter. Then we'll show how the math is applied to the physics.

3.1 Linear differential equations

A *linear differential equation* is one in which x and its time derivatives enter only through their first powers. An example is $3\ddot{x}+7\dot{x}+x=0$. An example of a nonlinear differential equation is $3\ddot{x}+7\dot{x}^2+x=0$.

If the right-hand side of the equation is zero, then we use the term *homogeneous* differential equation. If the right-hand side is some function of t (as in the case of $3\ddot{x}-4\dot{x}=9t^2-5$), then we use the term *inhomogeneous* differential equation. The goal of this chapter will be to learn how to solve these two types of equations. Linear differential equations come up again and again in physics, so we had better find a systematic method of solving them.

The techniques that we will need are best learned through examples, so let's solve a few differential equations, starting with some simple ones. Throughout this chapter, x will be understood to be a function of t. Hence, a dot will denote time differentiation.

Example 1 ($\dot{x}=ax$): This is a very simple differential equation. There are (at least) two ways to solve it.

First method: Separate variables to obtain $dx/x = a\,dt$, and then integrate to obtain $\ln x = at + c$. Exponentiate to obtain

$$x = Ae^{at}, \tag{3.1}$$

where $A \equiv e^c$ is a constant factor. A is determined by the value of x at, say, $t = 0$.

Second method: Guess an exponential solution, that is, one of the form $x = Ae^{\alpha t}$. Substitution then immediately gives $\alpha = a$. Hence, the solution is $x = Ae^{at}$. Note that we can't solve for A, due to the fact that the equation is homogeneous and linear in x. (Translation: the A cancels out.) A is determined from the initial condition.

This method may seem a bit silly. And somewhat cheap. But as we will see below, guessing these exponential functions (or sums of them) is actually the most general thing we can try, so the method is indeed quite general.

REMARK: Using this method, you may be concerned that although you have found one solution to the equation, you might have missed another one. But the general theory of differential equations says that a first-order linear equation has only one independent solution (we'll just accept this fact here). So if you find one solution, you know that you've found the whole thing. ♣

Example 2 ($\ddot{x} = ax$): If a is negative, then this equation describes the oscillatory motion of, say, a spring (about which we'll have much more to say later). If a is positive, then it describes exponentially growing or decaying motion. There are (at least) three ways to solve this equation.

First method: You can use the separation-of-variables method of Section 2.3 here, because our system is one where the force depends on only the position x. But this method is rather cumbersome (as you found if you did Exercise 2.5). It will certainly work, but in the case where our equation is a *linear* function of x, there is a much simpler method:

Second method: As in the first example above, guess a solution of the form $x(t) = Ae^{\alpha t}$, and then find out what α must be. (Again, we can't solve for A, since it cancels out.) Plugging $Ae^{\alpha t}$ into $\ddot{x} = ax$ gives $\alpha = \pm\sqrt{a}$. We have therefore found two solutions. The most general solution is an arbitrary linear combination of these,

$$x(t) = Ae^{\sqrt{a}\,t} + Be^{-\sqrt{a}\,t}. \tag{3.2}$$

A and B are determined from the initial conditions.

VERY IMPORTANT REMARK: The fact that the sum of two different solutions is again a solution to our equation (as you should check right now) is a monumentally important property of *linear* differential equations. This property does *not* hold for nonlinear differential equations, for example $\ddot{x}^2 = x$, because the act of squaring after adding the two solutions produces a cross-term which destroys the equality (as you should again check right now).

This property is called the *principle of superposition*. That is, superimposing two solutions yields another solution. This quality makes theories in physics that are governed by linear equations *much* easier to deal with than those that are governed by nonlinear ones. General Relativity, for example, is permeated with nonlinear equations, and solutions to most General Relativity systems are extremely difficult to come by.

For equations with one main condition
(Those linear), we give you permission
To take your solutions,
With firm resolutions,
And add them in superposition. ♣

Let's say a little more about the solution in eq. (3.2).

If a is negative, then let's define $a \equiv -\omega^2$, where ω is a real number. The solution now becomes $x(t) = Ae^{i\omega t} + Be^{-i\omega t}$. Using $e^{i\theta} = \cos\theta + i\sin\theta$, this can be written in terms of trig functions, if desired. Various ways of writing the solution are:

$$\begin{aligned} x(t) &= Ae^{i\omega t} + Be^{-i\omega t} \\ x(t) &= C\cos\omega t + D\sin\omega t, \\ x(t) &= E\cos(\omega t + \phi_1), \\ x(t) &= F\sin(\omega t + \phi_2). \end{aligned} \tag{3.3}$$

The various constants are related to each other; for example, $C = E\cos\phi_1$, and $D = -E\sin\phi_1$. Note that there are two free parameters in each of the above expressions for $x(t)$. These parameters are determined by the initial conditions (say, the position and speed at $t = 0$). Depending on the specifics of a given problem, one of the above forms will work better than the others.

If a is positive, then let's define $a \equiv \omega^2$, where ω is a real number. The solution now becomes $x(t) = Ae^{\omega t} + Be^{-\omega t}$. Using $e^{\theta} = \cosh\theta + \sinh\theta$, this can be written in terms of hyperbolic trig functions, if desired. Various ways of writing the solution are:

$$\begin{aligned} x(t) &= Ae^{\omega t} + Be^{-\omega t} \\ x(t) &= C\cosh\omega t + D\sinh\omega t, \\ x(t) &= E\cosh(\omega t + \phi_1), \\ x(t) &= F\sinh(\omega t + \phi_2). \end{aligned} \tag{3.4}$$

Again, the various constants are related to each other. If you're unfamiliar with the hyperbolic trig functions, a few facts are listed in Appendix A.

Remarks: Although the solution in eq. (3.2) is completely correct for both signs of a, it is generally more illuminating to write the negative-a solutions in either the trig form or the $e^{\pm i\omega t}$ exponential form where the i's are explicit.

Again, you may be concerned that although you have found two solutions to the equation, you might have missed others. But the general theory of differential equations says that our second-order linear equation has only two independent solutions. Therefore, having found two independent solutions, we know we've found them all. ♣

The usefulness of this method of guessing exponential solutions cannot be overemphasized. It may seem somewhat restrictive, but it works. The examples in the remainder of this chapter should convince you of this.

This is our method, essential,
For equations we solve, differential.
It gets the job done,
And it's even quite fun.
We just try a routine exponential.

Example 3 ($\ddot{x} + 2\gamma\dot{x} + ax = 0$): This will be our last mathematical example, then we'll get into some physics. As we will see later, this example pertains to a damped harmonic oscillator. We have put a factor of 2 in the coefficient of $\dot{x}$ here in order to make some later formulas look nicer.

Note that the force in this example, which is $-2\gamma\dot{x} - ax$ (times m), depends on both v and x, so our methods of Section 2.3 don't apply. This leaves us with only our method of guessing an exponential solution, $Ae^{\alpha t}$. Plugging this into the given equation, and cancelling the nonzero factor of $Ae^{\alpha t}$, yields

$$\alpha^2 + 2\gamma\alpha + a = 0. \tag{3.5}$$

The solutions for α are

$$-\gamma \pm \sqrt{\gamma^2 - a}. \tag{3.6}$$

Call these α_1 and α_2. Then the general solution to our equation is

$$\begin{aligned} x(t) &= Ae^{\alpha_1 t} + Be^{\alpha_2 t} \\ &= e^{-\gamma t}\left(Ae^{t\sqrt{\gamma^2 - a}} + Be^{-t\sqrt{\gamma^2 - a}}\right). \end{aligned} \tag{3.7}$$

(Well, our method of trying $Ae^{\alpha t}$ doesn't look so trivial anymore...)

If $\gamma^2 - a < 0$, then we can write our answer in terms of sines and cosines, so we have oscillatory motion which decreases in time due to the $e^{-\gamma t}$ factor (or it increases, if $\gamma < 0$, but this is rarely physical). If $\gamma^2 - a > 0$, then we have exponential motion. More on these different possibilities in Section 3.2.2.

In general, if we have a homogeneous linear differential equation of the form

$$\frac{d^n x}{dt^n} + c_{n-1}\frac{d^{n-1}x}{dt^{n-1}} + \cdots + c_1\frac{dx}{dt} + c_0 x = 0, \tag{3.8}$$

then our strategy is to guess an exponential solution, $x(t) = Ae^{\alpha t}$, and then (in theory) solve the resulting nth order equation (namely $\alpha^n + c_{n-1}\alpha^{n-1} + \cdots + c_1\alpha + c_0 = 0$), for α, to obtain the solutions $\alpha_1, \ldots, \alpha_n$. The general solution for $x(t)$ is then

$$x(t) = A_1 e^{\alpha_1 t} + A_2 e^{\alpha_2 t} + \cdots + A_n e^{\alpha_n t}, \tag{3.9}$$

where the A_i are determined by the initial conditions. In practice, however, we will rarely encounter differential equations of degree higher than 2. (Note: if some of the α_i happen to be equal, then eq. (3.9) is not valid. We will encounter such a situation in Section 3.2.2.)

3.2 Oscillatory motion

3.2.1 Simple harmonic motion

Let's now do some real live physical problems. We'll start with simple harmonic motion. This is the motion undergone by a particle subject to a force $F(x) = -kx$.

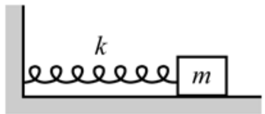

Figure 3.1

The classic system that undergoes simple harmonic motion is a mass attached to a spring (see Fig. 3.1). A typical spring has a force of the form $F(x) = -kx$, where x is the displacement from equilibrium. (This is "Hooke's law", and it holds as long as the spring isn't stretched too far; eventually this expression breaks down for any real spring.) Hence, $F = ma$ gives $-kx = m\ddot{x}$, or

$$\ddot{x} + \omega^2 x = 0, \qquad \text{where} \quad \omega \equiv \sqrt{\frac{k}{m}}. \tag{3.10}$$

From Example 2 in the previous section, the solution to this may be written as in eq. (3.3),

$$x(t) = A\cos(\omega t + \phi). \tag{3.11}$$

The system therefore oscillates back and forth forever in time.

Remark: The constants A and ϕ are determined by the initial conditions. If, for example, $x(0) = 0$ and $\dot{x}(0) = v$, then we must have $0 = A\cos\phi$ and $v = -A\omega\sin\phi$. Hence, $\phi = \pi/2$, and $A = -v/\omega$. Therefore, the solution is $x(t) = -(v/\omega)\cos(\omega t + \pi/2)$. This looks a little nicer as $x(t) = (v/\omega)\sin(\omega t)$. So, given these initial conditions, we could have arrived at this result a little quicker if we had chosen the "sin" solution in eq. (3.3). ♣

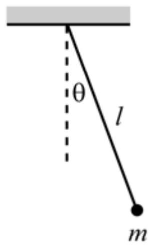

Figure 3.2

Example (Simple pendulum): Another classic system that undergoes (approximate) simple harmonic motion is the simple pendulum, that is, a mass that hangs on a massless string and swings in a vertical plane.

Let ℓ be the length of the string. Let θ be the angle the string makes with the vertical (see Fig. 3.2). Then the gravitational force on the mass in the tangential direction is $-mg\sin\theta$. So $F = ma$ in the tangential direction gives

$$-mg\sin\theta = m(\ell\ddot{\theta}) \tag{3.12}$$

(The tension in the string exactly cancels the radial component of gravity, so the radial $F = ma$ serves only to tell us the tension, which is of no use to us here.) We will now enter the realm of approximations and assume that the amplitude of the oscillations is small. (Without this approximation, the problem cannot be solved exactly.) This allows us to write $\sin\theta \approx \theta$, which gives

$$\ddot{\theta} + \omega^2\theta = 0, \qquad \text{where} \quad \omega \equiv \sqrt{\frac{g}{\ell}}. \tag{3.13}$$

Therefore,

$$\theta(t) = A\cos(\omega t + \phi), \tag{3.14}$$

where A and ϕ are determined from the initial conditions.

The true motion is arbitrarily close to this, for sufficiently small amplitudes. Exercise 3 deals with the higher-order corrections to the motion in the case where the amplitude is not small.

3.2.2 Damped harmonic motion

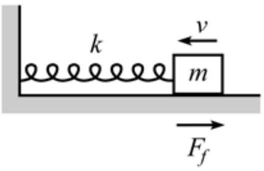

Figure 3.3

Consider a mass m attached to the end of a spring which has a spring constant k. Let the mass be subject to a drag force proportional to its velocity, $F_f = -bv$ (see Fig. 3.3). What is the position as a function of time?

The force on the mass is $F = -b\dot{x} - kx$. So $F = m\ddot{x}$ gives

$$\ddot{x} + 2\gamma\dot{x} + \omega^2 x = 0, \tag{3.15}$$

where $2\gamma \equiv b/m$, and $\omega \equiv \sqrt{k/m}$. But this is exactly the equation we solved in Example 3 in the previous section (with $a \to \omega^2$). Now, however, we have the physical restrictions that $\gamma > 0$ and $\omega^2 > 0$. Letting $\Omega^2 \equiv \gamma^2 - \omega^2$, for simplicity, we may write the solution in eq. (3.7) as

$$x(t) = e^{-\gamma t}\left(Ae^{\Omega t} + Be^{-\Omega t}\right), \qquad \text{where } \Omega^2 \equiv \gamma^2 - \omega^2. \tag{3.16}$$

There are three cases to consider.

Case 1: Underdamping ($\Omega^2 < 0$)

In this case, $\omega > \gamma$. Since Ω is imaginary, let us define $\Omega \equiv i\tilde{\omega}$ (so $\tilde{\omega} = \sqrt{\omega^2 - \gamma^2}$). Eq. (3.16) then gives

$$\begin{aligned} x(t) &= e^{-\gamma t}\left(Ae^{i\tilde{\omega}t} + Be^{-i\tilde{\omega}t}\right) \\ &\equiv e^{-\gamma t}C\cos(\tilde{\omega}t + \phi). \end{aligned} \tag{3.17}$$

These two forms are equivalent. Depending on the circumstances of the problem, one form works better than the other. (Or perhaps one of the other forms in eq. (3.3) will be the most useful one, to be multiplied by the $e^{-\gamma t}$ factor.) The constants are related by $A + B = C\cos\phi$ and $A - B = iC\sin\phi$. In a physical problem, $x(t)$ is real, so we must have $A^* = B$ (where the star denotes complex conjugation). The two constants A and B, or C and ϕ, are determined from the initial conditions.

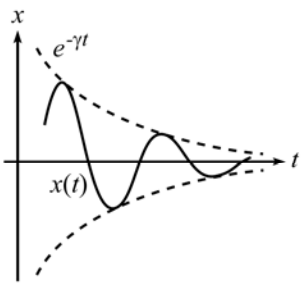

Figure 3.4

The cosine form makes it apparent that the motion is harmonic motion whose amplitude decreases in time because of the $e^{-\gamma t}$ factor. A plot of such motion is shown in Fig. 3.4. Note that the frequency of the motion, $\tilde{\omega} = \sqrt{\omega^2 - \gamma^2}$, is less than the natural frequency, ω, of the undamped oscillator.

REMARKS: If γ is very small, then $\tilde{\omega} \approx \omega$, which makes sense, because we almost have an undamped oscillator. If γ is very close to ω, then $\tilde{\omega} \approx 0$. So the oscillations are very slow. Of course, for very small $\tilde{\omega}$ it's hard to even tell that the oscillations exist, since they will damp out on a time scale of order $1/\gamma$, which will be short compared to the long time scale of the oscillations, $1/\tilde{\omega}$. ♣

Case 2: Overdamping ($\Omega^2 > 0$)

In this case, $\omega < \gamma$. Ω is real (and taken to be positive), so eq. (3.16) gives

$$x(t) = Ae^{-(\gamma-\Omega)t} + Be^{-(\gamma+\Omega)t}. \tag{3.18}$$

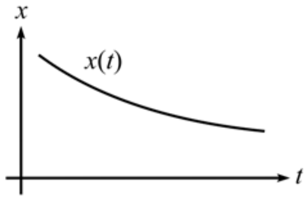

Figure 3.5

There is no oscillatory motion in this case (see Fig. 3.5). Note that $\gamma > \Omega \equiv \sqrt{\gamma^2 - \omega^2}$, so both of the exponents are negative. The motion therefore goes to zero for large t. (This had better be the case. A real spring is not going to have the motion go off to infinity. If we had obtained a positive exponent somehow, we'd know we had made a mistake.)

Remarks: If γ is just slightly larger than ω, then $\Omega \approx 0$, so the two terms in (3.18) are roughly equal, and we essentially have exponential decay, according to $e^{-\gamma t}$. If $\gamma \gg \omega$ (that is, strong damping), then $\Omega \approx \gamma$, so the first term in (3.18) dominates, and we essentially have exponential decay according to $e^{-(\gamma-\Omega)t}$. We can be somewhat quantitative about this by approximating Ω as $\Omega \equiv \sqrt{\gamma^2 - \omega^2} = \gamma\sqrt{1 - \omega^2/\gamma^2} \approx \gamma(1 - \omega^2/2\gamma^2)$. Therefore, the exponential behavior goes like $e^{-\omega^2 t/2\gamma}$. This is slow decay (that it, slow compared to $t \sim 1/\omega$), which makes sense if the damping is very strong. The mass slowly creeps back to the origin, as for a weak spring immersed in molasses. ♣

Case 3: Critical damping ($\Omega^2 = 0$)

In this case, $\gamma = \omega$. Eq. (3.15) therefore becomes $\ddot{x} + 2\gamma\dot{x} + \gamma^2 x = 0$. In this special case, we have to be careful in solving our differential equation. The solution in eq. (3.16) is not valid, because in the procedure leading to eq. (3.7), the roots α_1 and α_2 are equal (to $-\gamma$), so we have really found only one solution, $e^{-\gamma t}$. We'll just invoke here the result from the theory of differential equations which says that in this special case, the other solution is of the form $te^{-\gamma t}$.

Remark: You should check explicitly that $te^{-\gamma t}$ solves the equation $\ddot{x}+2\gamma\dot{x}+\gamma^2 x = 0$. Or if you want to, you can derive it in the spirit of Problem 1. In the more general case where there are n identical roots in the procedure leading to eq. (3.9) (call them all α), the n independent solutions to the differential equation are $t^k e^{\alpha t}$, for $0 \le k \le (n-1)$. But more often than not, there are no repeated roots, so you don't have to worry about this. ♣

Our solution is therefore of the form

$$x(t) = e^{-\gamma t}(A + Bt). \tag{3.19}$$

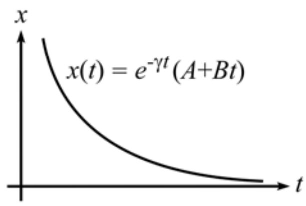

Figure 3.6

The exponential factor eventually wins out over the Bt term, of course, so the motion goes to zero for large t (see Fig. 3.6).

If we are given a spring with a fixed ω, and if we look at the system at different values of γ, then critical damping (when $\gamma = \omega$) is the case where the motion converges to zero in the quickest way (which is like $e^{-\omega t}$). This is true because in the underdamped case ($\gamma < \omega$), the envelope of the oscillatory motion goes like $e^{-\gamma t}$, which goes to zero slower than $e^{-\omega t}$, since $\gamma < \omega$. And in the overdamped case ($\gamma > \omega$), the dominant piece is the $e^{-(\gamma-\Omega)t}$ term. And as you can verify, if $\gamma > \omega$ then $\gamma - \Omega \equiv \gamma - \sqrt{\gamma^2 - \omega^2} < \omega$, so this motion also goes to zero slower than $e^{-\omega t}$.

Critical damping is very important in many real systems, such as screen doors and automobile shock absorbers, where the goal is to have the system head to zero (without overshooting and bouncing around) as fast as possible.

3.2.3 Driven (and damped) harmonic motion

Mathematical prelude

Before we examine driven harmonic motion, we have to learn how to solve a new type of differential equation. How can we solve something of the form

$$\ddot{x} + 2\gamma\dot{x} + ax = C_0 e^{i\omega_0 t}, \tag{3.20}$$

where γ, a, and ω_0 are given quantities? This is an inhomogeneous differential equation, due to the term on the right-hand side. It's not very physical, since the right-hand side is complex, but we're doing math now. Equations of this sort will come up again and again, and fortunately there is a nice, easy (although sometimes messy) method for solving them. As usual, the method is to make a reasonable guess, plug it in, and see what condition comes out.

Since we have the $e^{i\omega_0 t}$ sitting on the right side, let's try a solution of the form $x(t) = Ae^{i\omega_0 t}$. (A will depend on ω_0, among other things, as we will see.) Plugging this into eq. (3.20), and cancelling the non-zero factor of $e^{i\omega_0 t}$, we obtain

$$(-\omega_0^2)A + 2\gamma(i\omega_0)A + aA = C_0. \tag{3.21}$$

Solving for A, we find our solution for x to be

$$x(t) = \left(\frac{C_0}{-\omega_0^2 + 2i\gamma\omega_0 + a}\right) e^{i\omega_0 t}. \tag{3.22}$$

Note the differences between this technique and the one in Example 3 in Section 3.1. In that example, the goal was to determine what the α in $x(t) = Ae^{\alpha t}$ had to be. And there was no way to solve for A; the initial conditions determined A. But in the present technique, the ω_0 in $x(t) = Ae^{i\omega_0 t}$ is a given quantity, and the goal is to solve for A in terms of the given constants. Therefore, in the solution in eq. (3.22), there are *no free constants* to be determined by initial conditions. We've found one particular solution, and we're stuck with it. (The term *particular solution* is what people use for eq. (3.22).)

With no freedom to adjust the solution in eq. (3.22), how can we satisfy an arbitrary set of initial conditions? Fortunately, eq. (3.22) does not represent the most general solution to eq. (3.20). The most general solution is the sum of our particular solution in eq. (3.22), *plus* the "homogeneous" solution we found in eq. (3.7). This sum is certainly a solution, because the solution in eq. (3.7) was explicitly constructed to yield zero when plugged into the left-hand side of eq. (3.20). Therefore, tacking it onto our particular solution won't change the equality in eq. (3.20), because the left side is linear. The principle of superposition has saved the day.

The complete solution to eq. (3.20) is therefore

$$x(t) = e^{-\gamma t}\left(Ae^{t\sqrt{\gamma^2-a}} + Be^{-t\sqrt{\gamma^2-a}}\right) + \left(\frac{C_0}{-\omega_0^2 + 2i\gamma\omega_0 + a}\right) e^{i\omega_0 t}, \tag{3.23}$$

where A and B are determined by the initial conditions.

With superposition in mind, it is clear what the strategy should be if we have a slightly more general equation to solve, for example,

$$\ddot{x} + 2\gamma\dot{x} + ax = C_1 e^{i\omega_1 t} + C_2 e^{i\omega_2 t}. \tag{3.24}$$

Simply solve the equation with only the first term on the right. Then solve the equation with only the second term on the right. Then add the two solutions. And then add on the homogeneous solution from eq. (3.7). We are able to apply the principle of superposition because the left-hand side of our equation is linear.

Finally, let's look at the case where we have many such terms on the right-hand side, for example,

$$\ddot{x} + 2\gamma\dot{x} + ax = \sum_{n=1}^{N} C_n e^{i\omega_n t}. \tag{3.25}$$

We simply have to solve N different equations, each with just one of the N terms on the right-hand side. Then add up all the solutions, then add on the homogeneous solution from eq. (3.7). If N is infinite, that's fine. You'll just have to add up an infinite number of solutions. This is the principle of superposition at its best.

Remark: The previous paragraph, combined with a basic result from Fourier analysis, allows us to solve (in principle) any equation of the form

$$\ddot{x} + 2\gamma\dot{x} + ax = f(t). \tag{3.26}$$

Fourier analysis says that any (nice enough) function, $f(t)$, may be decomposed into its Fourier components,

$$f(t) = \int_{-\infty}^{\infty} g(\omega) e^{i\omega t} d\omega. \tag{3.27}$$

In this continuous sum, the functions $g(\omega)$ take the place of the coefficients C_n in eq. (3.25). So, if $S_\omega(t)$ is the solution for $x(t)$ when there is only the term $e^{i\omega t}$ on the right-hand side of eq. (3.26) (that is, $S_\omega(t)$ is the solution given in eq. 3.22)), then the complete particular solution to (3.26) is

$$x(t) = \int_{-\infty}^{\infty} g(\omega) S_\omega(t)\, d\omega. \tag{3.28}$$

Finding the coefficients $g(\omega)$ is the hard part (or, rather, the messy part), but we won't get into that here. We won't do anything with Fourier analysis in this book, but we just wanted to let you know that it *is* possible to solve (3.26) for any function $f(t)$. Most of the functions we'll consider will be nice functions like $\cos\omega_0 t$, for which the Fourier decomposition is simply the finite sum, $\cos\omega_0 t = \frac{1}{2}(e^{i\omega_0 t} + e^{-i\omega_0 t})$. ♣

Let's now do a physical example.

Example (Damped and driven spring): Consider a spring with spring constant k. A mass m at the end of the spring is subject to a friction force proportional to its velocity, $F_f = -bv$. The mass is also subject to a driving force, $F_d(t) = F_d \cos\omega_d t$ (see Fig. 3.7). What is its position as a function of time?

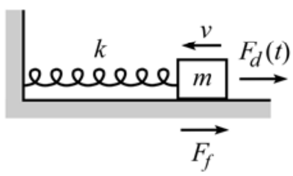

Figure 3.7

Solution: The force on the mass is $F(x, \dot{x}, t) = -b\dot{x} - kx + F_d \cos\omega_d t$. So $F = ma$ gives

$$\begin{aligned} \ddot{x} + 2\gamma\dot{x} + \omega^2 x &= F\cos\omega_d t \\ &= \frac{F}{2}\left(e^{i\omega_d t} + e^{-i\omega_d t}\right). \end{aligned} \tag{3.29}$$

where $2\gamma \equiv b/m$, $\omega \equiv \sqrt{k/m}$, and $F \equiv F_d/m$. Using eq. (3.22) and the technique of adding solutions mentioned after eq. (3.24), our particular solution is

$$x_p(t) = \left(\frac{F/2}{-\omega_d^2 + 2i\gamma\omega_d + \omega^2}\right)e^{i\omega_d t} + \left(\frac{F/2}{-\omega_d^2 - 2i\gamma\omega_d + \omega^2}\right)e^{-i\omega_d t}. \tag{3.30}$$

The complete solution is the sum of this particular solution and the homogeneous solution from eq. (3.16).

Let's simplify eq. (3.30) a bit. Getting the i's out of the denominators, and turning the exponentials into sines and cosines, we find (after a little work)

$$x_p(t) = \left(\frac{F(\omega^2 - \omega_d^2)}{(\omega^2 - \omega_d^2)^2 + 4\gamma^2\omega_d^2}\right)\cos\omega_d t + \left(\frac{2F\gamma\omega_d}{(\omega^2 - \omega_d^2)^2 + 4\gamma^2\omega_d^2}\right)\sin\omega_d t. \tag{3.31}$$

Note that this is real, as it must be, if it is to describe the position of a particle.

REMARKS: If you wish, you can solve eq. (3.29) simply by taking the real part of the solution to eq. (3.20), that is, the $x(t)$ in eq. (3.22). This is true because if we take the real part of eq. (3.20), we obtain

$$\frac{d^2}{dt^2}\big(\mathrm{Re}(x)\big) + 2\gamma\frac{d}{dt}\big(\mathrm{Re}(x)\big) + a\big(\mathrm{Re}(x)\big) = \mathrm{Re}\left(C_0 e^{i\omega_0 t}\right) = C_0\cos(\omega_0 t) \tag{3.32}$$

In other words, if x satisfies eq. (3.20) with a $C_0 e^{i\omega_0 t}$ on the right side, then $\mathrm{Re}(x)$ satisfies it with a $C_0\cos(\omega_0 t)$ on the right.

At any rate, it is clear that (with $C_0 = F$) the real part of eq. (3.22) does indeed give the result in eq. (3.31), because in eq. (3.30) we just took half of a quantity plus its complex conjugate, which is the real part.

If you don't like using complex numbers, another way of solving eq. (3.29) is to keep it in the form with the $\cos\omega_d t$ on the right, and then simply guess a solution of the form $A\cos\omega_d t + B\sin\omega_d t$, and solve for A and B (this is the task of Problem 5). The result will be eq. (3.31). ♣

We can simplify eq. (3.31) a bit further. If we define

$$R \equiv \sqrt{(\omega^2 - \omega_d^2)^2 + (2\gamma\omega_d)^2}, \tag{3.33}$$

then we may rewrite eq. (3.31) as

$$\begin{aligned} x_p(t) &= \frac{F}{R}\left(\frac{(\omega^2 - \omega_d^2)}{R}\cos\omega_d t + \frac{2\gamma\omega_d}{R}\sin\omega_d t\right) \\ &\equiv \frac{F}{R}\cos(\omega_d t - \phi), \end{aligned} \tag{3.34}$$

where ϕ is defined by

$$\cos\phi = \frac{\omega^2 - \omega_d^2}{R}, \quad \sin\phi = \frac{2\gamma\omega_d}{R} \qquad \Longrightarrow \qquad \tan\phi = \frac{2\gamma\omega_d}{\omega^2 - \omega_d^2}. \tag{3.35}$$

(Note that $0 < \phi < \pi$, since $\sin\phi$ is positive.)

Recalling the homogeneous solution in eq. (3.16), we may write the complete solution to eq. (3.29) as

$$x(t) = \frac{F}{R}\cos(\omega_d t - \phi) + e^{-\gamma t}\left(Ae^{\Omega t} + Be^{-\Omega t}\right). \tag{3.36}$$

The constants A and B are determined by the initial conditions. Note that if there is any damping at all in the system (that is, $\gamma > 0$), then the homogeneous part of the solution goes to zero for large t, and we are left with only the particular solution. In other words, the system approaches a definite $x(t)$, namely $x_p(t)$, independent of the initial conditions.

REMARK: The amplitude of the solution in eq. (3.34) is proportional to $1/R = [(\omega^2 - \omega_d^2)^2 + (2\gamma\omega_d)^2]^{-1/2}$. Given ω_d and γ, this is maximum when $\omega = \omega_d$. Given ω and γ, it is maximum when $\omega_d = \sqrt{\omega^2 - 2\gamma^2}$ (as you can show in Exercise 4); in the case of weak damping (that is, $\gamma \ll \omega$), the maximum is achieved when $\omega_d \approx \omega$. The term *resonance* is used to describe this situation where the amplitude of oscillations is as large as possible. Note that, using eq. (3.35), the phase angle ϕ equals $\pi/2$ when $\omega_d \approx \omega$. Hence, the motion of the particle lags the driving force by a quarter of a cycle at resonance. ♣

3.3 Coupled oscillators

Mathematical prelude

In the previous sections, we have dealt with only one function of time, $x(t)$. What if we have two functions of time, say $x(t)$ and $y(t)$, which are related by a pair of "coupled" differential equations? For example,

$$\begin{aligned} 2\ddot{x} + \omega^2(5x - 3y) &= 0, \\ 2\ddot{y} + \omega^2(5y - 3x) &= 0. \end{aligned} \tag{3.37}$$

We'll assume $\omega^2 > 0$ here, but this isn't necessary. We call these equations "coupled" because there are x's and y's in both of them, and it is not immediately obvious how to separate them to solve for x and y. There are (at least) two methods of solving these equations.

First method: Sometimes it is easy, as in this case, to find certain linear combinations of the given equations for which nice things happen. Taking the sum, we find

$$(\ddot{x} + \ddot{y}) + \omega^2(x + y) = 0. \tag{3.38}$$

This equation involves x and y only in the combination of their sum, $x + y$. With $z \equiv x + y$, it is just our old friend, $\ddot{z} + \omega^2 z = 0$. The solution is

$$x + y = A_1\cos(\omega t + \phi_1), \tag{3.39}$$