Complex numbers

In principle, there is really nothing that you have to remember about complex numbers except

$$i^2 = -1 \qquad I$$

Everything else about complex numbers follows from this plus algebra. i is like any other algebraic variable except that $i^2 = -1$. A complex number has the general form

$$a + ib$$

where a is the real part and b is called the imaginary part (**this is stupid — logically, the imaginary part should be called ib which is actually imaginary** — but the convention is that the imaginary part is the real number b — sorry about that). So for example if you want to multiply two complex numbers

$$(a + ib)(x + iy)$$

you just expand it as you would any algebraic expression

$$ax + ibx + iby + i^2by$$

then use I

$$ax + ibx + iby - by$$

then if you like, you can collect terms back again into the conventional form

$$(ax - by) + i(ay + bx)$$

However, it is useful to remember a few other definitions and relations.

If $z = a + ib$

The complex conjugate, z^***,** of the complex number z, is obtained by changing the sign of i:

$$z^* = a - ib\,.$$

Note that $\mathrm{Re}(z) = (z + z^*)/2$ and $\mathrm{Im}(z) = (z - z^*)/2i\,.$

If $z = a + ib$

The complex plane: Because a complex number z is specified by two real numbers, you can and should think of it as a two-dimensional vector, with components (a, b). The real part of z, $a = \mathrm{Re}(z)$, is the x component and the imaginary part of z, $b = \mathrm{Im}(z)$, is the y component.

If $z = a + ib$

The absolute value, $|z|$**,** of z, is the length of the vector (a, b):

$$|z| = \sqrt{a^2 + b^2} = \sqrt{z^* z}$$

The absolute value $|z|$ is always a real, non-negative number.

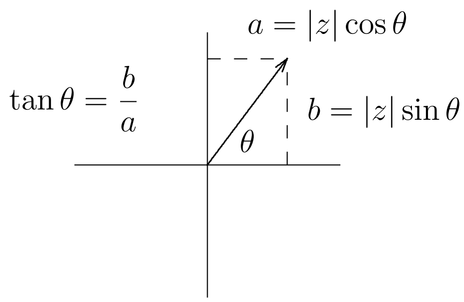
If $z = a + ib$

The argument or phase, $\arg(z)$**,** of a nonzero complex number z, is the angle, in radians, of the vector (a, b) counterclockwise from the x axis:

$$\arg(z) = \begin{cases} \arctan(b/a) \text{ for } a \geq 0\,, \\ \arctan(b/a) + \pi \text{ for } a < 0\,. \end{cases}$$

$\arg(z + 2n\pi) = \arg(z)$ for integer n

If $z = a + ib$**, then the vector** (a, b) **in polar coordinates** (R, θ)**, has** $R = |z|$ **and** $\theta = \arg(z)$



The harmonic oscillator

We can solve the equation of motion for the harmonic oscillator by guessing an exponential solution:

$$m\,\frac{d^2}{dt^2}z(t) = -K\,z(t) \tag{1}$$

$$z(t) \to e^{Ht} \tag{2}$$

$$m\,\frac{d^2}{dt^2}e^{Ht} = -K\,e^{Ht} \tag{3}$$

$$m\,H^2\,e^{Ht} = -K\,e^{Ht} \tag{4}$$

$$\left(m\,H^2 + K\right)\,e^{Ht} = 0 \tag{5}$$

$$m\,H^2 + K = 0 \tag{6}$$

$$H = \pm i\omega \quad \text{for} \quad i = \sqrt{-1} \quad \text{and} \quad \omega = \sqrt{\frac{K}{M}} \tag{7}$$

The connection with sines and cosines is Euler's formula, one of the more amusing relations in mathematics:

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{8}$$

There are many ways of seeing this. Let's just use the Taylor expansion

$$e^{i\theta} = 1 + (i\theta) + \frac{1}{2}(i\theta)^2 + \frac{1}{3!}(i\theta)^3 + \frac{1}{4!}(i\theta)^4 + \cdots \tag{9}$$

$$= \left(1 - \frac{1}{2}\theta^2 + \frac{1}{4!}(\theta)^4 + \cdots\right) + i\left(\theta - \frac{1}{3!}\theta^3 + \cdots\right) \tag{10}$$

Since both simes of (8) have the same Taylor expansion, the two sides must be equal.

Euler's formula is the connection between algebra and trigonometry! You can define the trigonometric functions this way:

$$\cos\theta \equiv \frac{e^{i\theta} + e^{-i\theta}}{2} \tag{11}$$

$$\sin\theta \equiv \frac{e^{i\theta} - e^{-i\theta}}{2i} \tag{12}$$

Now you can derive all trigonometric identities just using algebra.

These are equivalent: —

$$x(t) = a\cos\omega t + b\sin\omega t$$

$$\text{and}$$

$$x(t) = c\,e^{i\omega t} + d\,e^{-i\omega t} \tag{13}$$

$$= c\,(\cos\omega t + i\sin\omega t) + d\,(\cos\omega t - i\sin\omega t)$$

$$a = c + d \qquad b = i(c - d)$$

To make a and b real you should take $d = c^*$

There is an important general principle at work here. The equation of motion for the harmonic oscillator is linear in x. Linearity guarantees that a linear combination of two possible trajectories is another possible trajectory. This is what allows us to write the most general solution as a combination of the two complex exponential solutions times constants:

$$x(t) = c\,e^{i\omega t} + d\,e^{-i\omega t} \tag{14}$$

Uniform circular motion

One very evocative way to think about these complex solutions is in what is called "the complex plane." Because a complex number has two real components, its real and its imaginary part, we can think of a complex number as a real vector in a two dimensional space. This two dimensional space is the complex plane. In the complex plane, the basis solution $e^{i\omega t}$ has real part $\cos\omega t$ and imaginary part $\sin\omega t$, so its counterpart in the complex plane is the two dimensional vector, $(\cos\omega t, \sin\omega t)$,

$$e^{i\omega t} = \cos\omega t + i\sin\omega t \to (\cos\omega t, \sin\omega t) \tag{15}$$

But this is a unit vector an angle ωt from the x axis. Thus as t increases, $e^{i\omega t}$ executes uniform circular motion in the complex plane. You can see this in the DOS program UNIFORM.EXE (press F10 to quit). More generally, a complex number $z = x + iy$ and be written equivalently as a positive number R times a complex exponential $e^{i\theta}$. Note the connection of this with the relation between Cartesian and Polar coordinates in the complex plane.

$$z = x + iy = R\,e^{i\theta} \rightarrow (x, y)_{\text{Cartesian}} \Leftrightarrow (R, \theta)_{\text{Polar}} \tag{16}$$

$$R = |z| = \sqrt{x^2 + y^2} \tag{17}$$

$$\theta = \arg(z) \tag{18}$$

$$= \begin{cases} \arctan(b/a) \text{ for } a \geq 0\,, \\ \arctan(b/a) + \pi \text{ for } a < 0\,. \end{cases} \tag{19}$$

The damped harmonic oscillation

Now include a frictional force of the form

$$-\gamma\, v \tag{20}$$

with

$$\gamma \geq 0 \tag{21}$$

After our discussion of the harmonic oscillator, you will notice, I hope that this force law is also linear. There is just a single factor of x in

$$v = \frac{dx}{dt} \tag{22}$$

Let's add this kind of friction to our harmonic oscillator. With such a frictional term, the equation of motion for the mass on a spring becomes

$$m\frac{d^2}{dt^2}\,x(t) = -\gamma\,\frac{d}{dt}\,x(t) - K\,x(t) \tag{23}$$

$$\frac{d^2}{dt^2}\,x(t) + \frac{\gamma}{m}\,\frac{d}{dt}\,x(t) + \omega_0^2\,x(t) = 0 \tag{24}$$

where $\omega_0 = \sqrt{K/m}$. Again let's try an exponential solution. This will allow us to write down an explicit solution and understand it rather simply.

Because we suspect that we may need to consider complex solutions to make them look simple, we will consider a complex solution $z(t)$ whose real part is the real solution in which we are ultimately interested.

$$x(t) = \operatorname{Re} z(t) \tag{25}$$

Because of linearity, the equation of motion for this complex solution looks the same as for the real solution.

$$\frac{d^2}{dt^2}\,z(t) + \Gamma\,\frac{d}{dt}\,z(t) + \omega_0^2\,z(t) = 0 \tag{26}$$

where

$$\Gamma = \gamma/m \tag{27}$$

But now because of time translation invariance and linearity, we can look for exponential solutions:

$$z(t) = e^{\alpha t} \tag{28}$$

$$\frac{d^2}{dt^2}\,e^{\alpha t} + \Gamma\,\frac{d}{dt}\,e^{\alpha t} + \omega_0^2\,e^{\alpha t} = 0 \tag{29}$$

As usual, derivatives with respect to t just bring down factors of α, so we can convert this to an algebraic equation:

$$(\alpha^2 + \Gamma\alpha + \omega_0^2)\,e^{\alpha t} = 0 \tag{30}$$

This has two solutions:

$$\alpha = -\frac{\Gamma}{2} \pm \sqrt{\frac{\Gamma^2}{4} - \omega_0^2} \equiv \Gamma_\pm \tag{31}$$

where

$$\Gamma > 0 \Rightarrow \mathrm{Re}\,\alpha > 0 \tag{32}$$

$z(t)$ oscillates if $\omega_0^2 > \Gamma^2/4$ and it just dies out if $\omega^2 < \Gamma^2/4$ —

$$z(t) \propto e^{\Gamma_\pm t} = \overbrace{e^{-\Gamma t/2}}^{\downarrow\text{ with }t} \times \underbrace{\overbrace{e^{\pm t\sqrt{\Gamma^2/4-\omega_0^2}}}^{\downarrow\uparrow\text{ if }\Gamma/2>\omega_0}}_{\bigcirc\text{ if }\Gamma/2<\omega_0} \tag{33}$$

The general solution is, as usual for a harmonic oscillator, a linear combination of the simple exponential basis solutions:

$$x(t) = b_+\,e^{\Gamma_+ t} + b_-\,e^{\Gamma_- t} \tag{34}$$

where $\Gamma_\pm$ are defined in (31). The constants $b_\pm$ contain the information about the initial conditions, just as in the undamped harmonic oscillator.

The nature of the trajectories described by (34) depends on the relative size of the two parameters, Γ and ω_0. If $\Gamma/2 > \omega_0$, the damping is large (this is called “overdamped”). In this case, both Γ_+ and Γ_- are real and negative, and the trajectory is a sum of decaying exponentials.

If $\Gamma/2 < \omega_0$, the damping is small (this is called “underdamped”). In this case, both Γ_+ and Γ_- have a negative real part and an imaginary part (with opposites signs). In this case, the trajectory oscillates (or circles in the complex plane), but also dies out with time exponentially in t — the DOS program DAMPED.EXE shows the oscillating case (again, press F10 to exit). Other kinds of damping would lead to other forms and more complicated dependence on the initial conditions, but the motion always dies out eventually if there is damping.

Forced oscillation and resonance

Suppose we "drive" our damped harmonic oscillator by adding a time dependent force, so the equation of motion becomes

$$\left(\frac{d^2}{dt^2} + \Gamma\,\frac{d}{dt} + \omega_0^2\right)\,x(t) = F(t)/m \tag{35}$$

Let's begin by discussing the linearity of this equation of motion. Because of the force term, the situation is a bit different from that of an unforced oscillator. Suppose that I have a solution to this equation, $x_1(t)$ and another one $x_2(t)$.

Now if I add them together, I don't get a solution to the same differential equation

$$\begin{gathered}
\left(\frac{d^2}{dt^2} + \Gamma\,\frac{d}{dt} + \omega_0^2\right)\,x_1(t) = F_1(t)/m \\
\left(\frac{d^2}{dt^2} + \Gamma\,\frac{d}{dt} + \omega_0^2\right)\,x_2(t) = F_2(t)/m \\
\Rightarrow \\
\left(\frac{d^2}{dt^2} + \Gamma\,\frac{d}{dt} + \omega_0^2\right)\,[a x_1(t) + b x_2(t)] \\
= [aF_1(t) + bF_2(t)]/m
\end{gathered} \tag{36}$$

In words, which are not very useful in this case, when we take linear combinations of the solutions, we must also take the same linear combinations of the driving forces, and vice versa.

In particular, this means that can always add a solution of the homogeneous equation, with no external force.

$$\begin{gathered}
\left(\frac{d^2}{dt^2} + \Gamma\,\frac{d}{dt} + \omega_0^2\right)\,x_1(t) = F_1(t)/m \\
\left(\frac{d^2}{dt^2} + \Gamma\,\frac{d}{dt} + \omega_0^2\right)\,x_0(t) = 0 \\
\Rightarrow \\
\left(\frac{d^2}{dt^2} + \Gamma\,\frac{d}{dt} + \omega_0^2\right)\,[a x_1(t) + b x_0(t)] = aF_1(t)/m
\end{gathered} \tag{37}$$

We can use this form of linearity to simplify the problem.

Harmonic driving forces

Life is much simpler if we look at forces of the form[1]

$$F(t) = F_0\,e^{-i\omega_d t} \tag{38}$$

[1]We can actually use linearity to write any force as a linear combination of forces of this form using the mathematical technique of Fourier analysis. So if we solve the problem for all values of ω_d, we have actually solved it for all reasonable forces.

where ω_d is called the "driving frequency." Why is this an interesting thing to do? This force is exponential — and therefore behaves very simply under time translations.

$$F(t+a) = e^{-i\omega_d a}\, F(t) \tag{39}$$

Thus we can look for solutions that are proportional to $e^{-i\omega_d t}$. There will also be terms in the general solution which are just like those we found for the undriven oscillator. We can always add these solutions because they do not contribute to the driving term. Thus the general solution will be of the form

$$z(t) = \mathcal{A}\, e^{-i\omega_d t} + b_+ e^{\Gamma_+ t} + b_- e^{\Gamma_- t} \tag{40}$$

where $\Gamma_\pm$ are defined in (31), so that

$$e^{\Gamma_\pm t}$$

are exponential solutions to the unforced oscillator problem. Just as in the case without damping, the coefficients of the "homogeneous" solutions must be set by initial conditions, but that is not true for $\mathcal{A}$. It does not depend on initial conditions! This is what is called a "particular" solution to the differential equation, and the general solution will also involve solutions to the equation without any force, and the coefficients of the solutions will have to be determined by the initial conditions.

One nice thing about the form (40) is that in certain cases, we do not care about the initial conditions. This is because the homogeneous solutions die out exponentially so long as there is any damping at all. If we wait long enough, only the term proportional to $\mathcal{A}$ survives.

Now let's compute $\mathcal{A}$. This is straightforward because we are working with exponentials.

$$\left(\frac{d^2}{dt^2} + \Gamma\,\frac{d}{dt} + \omega_0^2\right) \mathcal{A}\, e^{-i\omega_d t} = F_0\, e^{-i\omega_d t}/m$$

$$(-\omega_d^2 - i\Gamma\omega_d + \omega_0^2)\,\mathcal{A} = \frac{F_0}{m} \tag{41}$$

$$\mathcal{A} = \frac{F_0/m}{\omega_0^2 - i\Gamma\omega_d - \omega_d^2} \tag{42}$$

$$\mathcal{A} = \frac{F_0/m}{\omega_0^2 - \omega_d^2 - i\Gamma\omega_d} \tag{43}$$

$$\mathcal{A} = \frac{F_0/m}{\omega_0^2 - \omega_d^2 - i\Gamma\omega_d} \tag{44}$$

$$\mathcal{A} = \left(\frac{F_0/m}{\omega_0^2 - \omega_d^2 - i\Gamma\omega_d}\right)\left(\frac{\omega_0^2 - \omega_d^2 + i\Gamma\omega_d}{\omega_0^2 - \omega_d^2 + i\Gamma\omega_d}\right) \tag{45}$$

$$\mathcal{A} = \frac{(\omega_0^2 - \omega_d^2 + i\Gamma\omega_d)\, F_0/m}{(\omega_0^2 - \omega_d^2)^2 + \Gamma^2\omega_d^2} \tag{46}$$

Because we used the exponential solution, we got the solution just using algebra. It's $\propto F_0$.

$$\mathcal{A} = \frac{\left(\omega_0^2 - \omega_d^2 + i\Gamma\omega_d\right) F_0/m}{\left(\omega_0^2 - \omega_d^2\right)^2 + \Gamma^2\omega_d^2} = A + iB \tag{47}$$

$$A = \frac{\left(\omega_0^2 - \omega_d^2\right) F_0/m}{\left(\omega_0^2 - \omega_d^2\right)^2 + \Gamma^2\omega_d^2} \tag{48}$$

$$B = \frac{\Gamma\omega_d\, F_0/m}{\left(\omega_0^2 - \omega_d^2\right)^2 + \Gamma^2\omega_d^2} \tag{49}$$

Taking the real part gives

$$x(t) = \mathrm{Re}\left(\mathcal{A}e^{-i\omega_d t}\right) = A\cos\omega_d t + B\sin\omega_d t \tag{50}$$

Note the phase relations. The first term is in phase with the force if $\omega_0^2 > \omega_d^2$, that is when the system is driven slowly. In this limit, inertia is irrelevant, and the mass just moves along with the driving force.

If $\omega_0^2 < \omega_d^2$, when the system is driven rapidly, the first term is 180° out of phase with the force. In this limit, inertia dominates.

In between, for $\omega_0^2 = \omega_d^2$, the second term is crucial. It is 90° out of phase (behind) the driving force. You can play with these relations in the DOS program RESONATE.EXE, which animates driven damped oscillator. The up and down arrow keys change ω_d/ω_0, and again, press F10 to exit.

A and B are called the elastic and absorptive amplitudes. To see why, consider the work done by the driving force. We will consider work and energy more generally starting in the next lecture. For now, just use the result that you should remember from previous physics courses that the power, $P(t)$, which is the work per unit time done by the driving force, is the force times the velocity of the system on which it acts —

$$P(t) = F(t)\,\dot{x}(t) = F_0\cos\omega_d t \cdot \frac{\partial}{\partial t}(A\cos\omega_d t + B\sin\omega_d t) \tag{51}$$

Note that the power is a nonlinear function. For example, if F_0 doubles, both $F(t)$ and $\dot{x}(t)$ double, so the power quadruples. Because of this nonlinearity, we cannot use the complex form for $x(t)$ because we could get contributions to the power from both the real and imaginary part, which is not what we want physically. We must use real form for $x(t)$, which is what we have done. Continuing,

$$= P(t) = F_0\cos\omega_d t \cdot (-\omega_d\, A\sin\omega_d t + \omega_d\, B\cos\omega_d t) \tag{52}$$

$$= -F_0\omega_d A\cos\omega_d t\sin\omega_d t + F_0\omega_d B\cos^2\omega_d t \tag{53}$$

The first term in (53) averages to zero over a complete half-period of oscillation because $\cos\omega_d t\sin\omega_d t = \frac{1}{2}\sin 2\omega_d t$

$$\int_{t_0}^{t_0+\pi/\omega_d} dt\ \sin 2\omega_d t = -\frac{1}{2}\ \cos 2\omega_d t\Big|_{t_0}^{t_0+\pi/\omega_d} = 0 \tag{54}$$

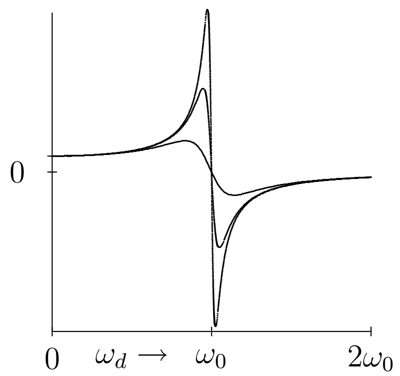
This is why A is called the elastic amplitude. If A dominates, energy that goes in comes back out, like an elastic collision in mechanics.

The second term is always positive — it averages to

$$P_{\text{average}} = \frac{1}{2} F_0 \omega_d B \quad (55)$$

B is called the absorptive amplitude because it measures how fast energy is absorbed by the system. P_{average}, is maximum on resonance, at $\omega_0 = \omega_d$. Furthermore, if the damping is small, the peak is very very sharp. This is one good way to find resonances.

Here are graphs of A and B for $\Gamma/\omega_d = 0.3$, 0.1 and 0.05 For larger values of Γ/ω_d, the resonance hardly shows up at all.



Each of these starts at

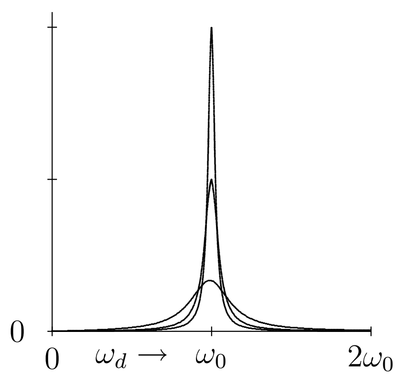
$$\frac{F_0}{m\omega_0^2} \quad (56)$$

for small damping, and looks like

$$\frac{F_0/m}{\omega_0^2 - \omega_d^2} \quad (57)$$

except near resonance.

Now, look at B for the same three values of Γ



Notice that as the Γ/ω_0 decreases, the resonance gets sharper and the effect of the B term is more and more concentrated near the resonance.